THE STABILITY OF PERIODIC MOTIONS (*)

(OB USTOICHIVOSTI PERIODICHESKIKH DVIZHENII)

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This paper is concerned with the stability of the integrals of differential equations of the form

$$x_s^{\cdot} = \sum_{k=1} a_{sk}x_k + X_s(x_1,\ldots,x_n;t) \qquad (s = 1,\ldots,n) \tag{0.1}$$

where a_{sk} are continuous periodic functions of t with a common basic period 2π. The functions X_s appear as series expansions in the variables $x_1, \ldots, x_n$ with periodic coefficients with the same period 2π.

Usually the problem of stability of periodic motions is considered as being different from the problem of the stability of equilibrium or that of a steady-state motion. For instance, this is the manner used in the works of Poincaré and Liapunov to analyze that question.

Here, it is proved that except for some nonessentially singular cases, the problem on the stability of periodic motions is always related to that of the stability of equilibrium.

Liapunov has proved that this proposition is valid for the case of linear systems, i.e. $X_s(x_1, \ldots, x_n;t) \equiv 0$, by transforming the system of linear differential equations with periodic coefficients into a system with constant coefficients [1].

1. Let us consider the general case when in the system (0.1) we have

$$X_s = \sum_{l\geqslant 2}^{\infty} X_s^{(l)}, \qquad X_s^{(l)} = \sum C_s^{(k_1\ldots k_n)}(t)\, x_1^{k_1}\ldots x_n^{k_n}$$

$$C_s^{(k_1\ldots k_n)}(t) = \sum_{p=-\infty}^{\infty} c_{sp}^{(k_1\ldots k_n)} e^{ipt} = \sum_{p=0}^{\infty} (a_{sp}^{(k_1\ldots k_n)} \cos pt + b_{sp}^{(k_1\ldots k_n)} \sin pt)$$

Using Liapunov's transformation, we can always bring the system (0.1) in a form for which all the coefficents of the linear parts are constants [1].

If the characteristic equation of the system (0.1) has m roots equal to one, q pairs of conjugate roots of modulus one (of the form $\nu_s = e^{\pm 2\pi i\lambda_s}$) and p roots with moduli smaller than one, then the determining equation of the transformed system has m roots equal to zero, q pairs of pure imaginary roots (of the form $\pm i\lambda_s$) and p roots with negative real parts. In that general case, the system of equations (0.1) can be represented in the form

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*) This work was received when the author was still alive: he took an active part in its preparation for printing. The reading of the proof pages during the printing process was done by V. G. Veretennikov.

$$
\begin{aligned}
y_s^{\cdot} &= \sum_{k=1}^{n_1} g_{sk} y_k + Y_s(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t) \\
z_j^{\cdot} &= \sum_{i=1}^{p} p_{ji} z_i + Z_j(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t)
\end{aligned} \tag{1.1}
$$

$$(s = 1, \ldots, n_1;\ n_1 = m + 2q;\ j = 1, \ldots, p)$$

Let us represent the functions Y_s and Z_j of the system (1. 1) by

$$
Y_s = Y_s^{(0)}(y_1, \ldots, y_{n_1}; t) + \sum_{K=1}^{\infty} P_s^*(z_1, \ldots, z_p; t)\, y_1^{k_1} \ldots y_{n_1}^{k_{n_1}} + \\
+ Y_{s1}(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t) \tag{1.2}
$$

$$
Z_j = Z_j^{(0)}(y_1, \ldots, y_{n_1}; t) + \sum_{K=1}^{\infty} Q_i^*(z_1, \ldots, z_p; t)\, y_1^{k_1} \ldots y_{n_1}^{k_{n_1}} + \\
+ Z_{j1}(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t) \qquad (K = k_1 + \ldots + k_{n_1})
$$

$$
Y_s^{(0)} = \sum_{k \geqslant 2}^{\infty} Y_s^{(k)}(y_1, \ldots, y_{n_1}; t), \qquad Z_j^{(0)} = \sum_{k \geqslant 2}^{\infty} Z_j^{(k)}(y_1, \ldots, y_{n_1}; t)
$$

Here P_s^* and Q_j^* are linear forms of the variables $z_1, \ldots, z_p$; in Sections 1 and 2 the superscript asterisk * replaces the index $k_1, \ldots, k_{n_1}$.

Equation $|g_{sk} - \delta_{sk}\nu| = 0$ has roots with zero real parts, and Equation

$$| p_{ji} - \delta_{ji}\varkappa | = 0$$

with negative real parts.

We shall assume, that the right-hand sides of the system (1. 1) satisfy the following conditions ;

1) The forms $Z_j^{(k)} \equiv 0$ for $k \leq N$.
2) The linear forms
$$P_s^{(k_1 \ldots k_{n_1})} \equiv 0 \quad \text{for} \quad k_1 + \ldots + k_{n_1} \leqslant N$$
3) The forms $Y_s^{(k)}$ for $k \leq N$ have constant coefficients

If these conditions are satisfied for the system

$$
y_s^{\cdot} = \sum_{k=1}^{n_1} g_{sk} y_k + \sum_{k \geqslant 2}^{N} Y_s^{(k)}(y_1, \ldots, y_{n_1}) \qquad (s = 1, \ldots, n_1) \tag{1.3}
$$

and if a Liapunov or Chetaev function is found, such that the sign of the derivatives of these functions is determined by forms of order not higher than the Nth and does not depend on the forms of higher order, then the corresponding functions for the complete system are determined in the form

$$
V = V_1(y_1, \ldots, y_{n_1}) + V_2(z_1, \ldots, z_p) \tag{1.4}
$$

where V_1 is the Liapunov or Chetaev function for the system (1. 3) and V_2 is determined from Equation

$$
\sum_{j=1}^{p} \frac{\partial V_2}{\partial z_j}(p_{j1} z_1 + \ldots + p_{jp} z_p) = M(z_1^2 + \ldots + z_p^2) \tag{1.5}
$$

Let us prove this assertion.

Let $V_1(y_1, \ldots, y_{n_1})$ be a positive definite function, satisfying the system of equations (1.3), and let its derivative V_1' be negative definite. Let us assume that the terms of order higher than N do not influence the sign of the derivative. We shall choose a function V_2 from (1.5), considering the quantity $M < 0$.

On the basis of Equations (1.1) when the conditions (1), (2) and (3) are satisfied, we can represent the derivative of the function V in the form

$$V' = V_1'(y_1, \ldots, y_{n_1}) + M(z_1^2 + \ldots + z_p^2) +$$

$$+ \sum_{s=1}^{n_1} \frac{\partial V_1}{\partial y_s} \Big[\sum_{k=N+1}^{\infty} Y_s^{(k)}(y_1, \ldots, y_{n_1}; t) + \sum_{K=N+1}^{\infty} P_s^*(z_1, \ldots, z_p; t)\, y_1^{k_1} \ldots y_{n_1}^{k_{n_1}} \Big] +$$

$$+ \sum_{s=1}^{n_1} \frac{\partial V_1}{\partial y_s} Y_{s1}(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t) + \sum_{j=1}^{p} \frac{\partial V_2}{\partial z_j} \Big[\sum_{k=N+1}^{\infty} Z_j^{(k)}(y_1, \ldots, y_{n_1}; t) +$$

$$+ \sum_{K=1}^{\infty} Q_j^*(z_1, \ldots, z_p; t)\, y_1^{k_1} \ldots y_{n_1}^{k_{n_1}} + Z_{j1}(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t) \Big] \qquad (1.6)$$

$$(K = k_1 + \ldots + k_{n_1})$$

This expression can be given the form

$$V' = V_1'(y_1, \ldots, y_{n_1}) + M(z_1^2 + \ldots + z_p^2) + \Xi^{(1)} + \Xi^{(2)}$$

$$\Xi^{(1)} = \sum_{K=N+1}^{\infty} R^*(z_1, \ldots, z_p; t)\, y_1^{k_1} \ldots y_{n_1}^{k_{n_1}}$$

$$\Xi^{(2)} = \sum_{i=1}^{p} \sum_{j=1}^{p} z_i z_j L_{ij}(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t) \qquad (K = k_1 + \ldots + k_{n_1})$$

For sufficiently small values of y_s, z_j the sign of the derivative V' is determined by the sign of Expression $M(z_1^2 + \ldots + z_p^2)$ independently from $\Xi^{(2)}$. Expression $\Xi^{(1)}$ does not change the sign of V_1'. Thus the sign of V' is determined by the sign of Expression

$$V_1' + M(z_1^2 + \ldots + z_p^2) < 0$$

The function $V(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p)$ (1.4) is positive definite. Consequently, the integrals of the system (1.1) are asymptotically stable if its right-hand sides satisfy the conditions (1) to (3).

Let us now assume that the system (1.3) is such that there corresponds to it a function of Chetaev $V_1(y_1, \ldots, y_{n_1})$.

Then the domain $V_1' > 0$ is enclosed inside the domain $V_1' > 0$, and this property of the function V_1' is determined by forms, the order of which is smaller than, or equal to N, independently from the forms of higher order.

We shall take a Chetaev function, satisfying the system (1.1), in the form

$$V = V_1(y_1, \ldots, y_{n_1}) + V_2(z_1, \ldots, z_p)$$

Here $V_2(z_1, \ldots, z_p)$ is determined from Equation (1.5) for $M > 0$.

On the basis of Equations (1.1) and when the conditions (1) to (3) are satisfied, the derivative of the function V can be represented in the form (1.6). Unlike in the previous case, the functions L_{ij} do not have to be equal to zero for $y_1 = \ldots = y_{n_1} = z_1 = \ldots = z_p = 0$, since the function V_1 can include linear terms. Denoting the values of the function L_{ij} by $L_{ij}^{(0)}$ for values of y_s, z_j equal to zero, we shall determine the number $M > 0$ such that Expression

$$U(z_1, \ldots, z_p) = M(z_1^2 + \ldots + z_p^2) + \sum_{i=1}^{p} \sum_{j=1}^{p} z_i z_j L_{ij}^{(0)}$$

represent a positive definite quadratic form . Then

$$V' = V_1'(y_1, \ldots, y_{n_1}) + U(z_1, \ldots, z_p) + \Xi^{(1)} + \Xi^{(3)}$$

$$\Xi^{(3)} = \sum_{i=1}^{p} \sum_{j=1}^{p} z_i z_j L_{ij}^{(1)}(y_1, \ldots, y_{n_1}; z_1, \ldots, z_p; t)$$

where $L_{ij}^{(1)}$ become equal to zero for $y_1 = \ldots = y_{n_1} = z_1 = \ldots = z_p = 0$. It is obvious that Expression $\Xi^{(3)}$ does not change the sign of $U(z_1, \ldots, z_p)$ for sufficiently small y_s, z_j, and $\Xi^{(1)}$ does not change the sign of V_1'. Consequently, in the domain $V > 0$, the function V' represents a sign-definite function of all variables y_s, z_j.. Since the function $V_2 < 0$, the domain $V > 0$ is enclosed inside the domain $V' > 0$. Thus, V is a Chetaev function for the system of equations (1.1). Similarly we can design functions V and W satisfying Chetaev's theorem for the system (1.1) [2].

2. We shall prove now that any system of the form (1.1) can be transformed into a new system, the right-hand side of which satisfy the conditions (1) to (3). The problems of stability with respect to the variables of the system (1.1) and the variables of the transformed system will be equivalent.

Let us introduce the substitution

$$z_j = \zeta_j + v_j(y_1, \ldots, y_{n_1}; t) \qquad (j = 1, \ldots, p;\ n_1 = m + 2q) \tag{2.1}$$

where v_j represent the N first terms of the series u_j which satisfy Equations

$$\frac{\partial u_j}{\partial t} + \sum_{s=1}^{n_1} \frac{\partial u_j}{\partial y_s}[g_{s1}y_1 + \ldots + g_{sn_1}y_{n_1} + Y_s(y_1, \ldots, y_{n_1}; u_1, \ldots, u_p; t)] =$$
$$= \sum_{i=1}^{p} p_{ji}u_i + Z_j(y_1, \ldots, y_{n_1}; u_1, \ldots, u_p; t) \tag{2.2}$$

In the general case the series u_j diverge.

We shall consider two possible cases.

In the first case, the substitution of the variables z_j in the Expressions (2.1) transforms exactly into zero all forms $Y_s^{(k)}(y_1, \ldots, y_{n_1}; t)$ corresponding to the values $k \leq N + 1$, no matter how large the number N is chosen.

This is possible only if $\quad Y_s(y_1, \ldots, y_{n_1}; u_1, \ldots, u_p; t) \equiv 0 \qquad (2.3)$

where u_j are series satisfying the system (2.2). That case is essentially singular. When investigating it, we shall consider the transformation

$$z_j = \zeta_j + u_j(y_1, \ldots, y_{n_1}; t) \qquad (j = 1, \ldots, p;\ n_1 = m + 2q)$$

This transformation is possible only when the series determined by Equations (2.2) converge. We shall prove the following.

Theorem 2.1. If the system (1.1) is such that:

1° the relation $|g_{sk} - \delta_{sk}\nu| = 0$ does not have any multiple roots, or if they occur, to each such root corresponds a number of solutions equal to its order of multiplicity :

2° there is no relation of the type

$$\sum_{s=1}^{n_1} m_s \nu_s - \varkappa_j = iE, \qquad i = \sqrt{-1} \qquad (j = 1, \ldots, p)$$

between the roots of Equations $|p_{ji} - \delta_{ji}\varkappa| = 0$ and $|g_{sk} - \delta_{sk}\nu| = 0$ where E is any integer including zero, and the m_s are positive integers satisfying the condition $m_1 + \ldots + m_{n_1} > 1$.

3° the functions $Y_s[y_1, \ldots, y_{n_1}; z_1(y_1, \ldots, y_{n_1}; t), \ldots, z_p(y_1, \ldots, y_{n_1}; t), t] \equiv 0$.

Then there exists a unique system of holomorphic functions $z_j = z_j(y_1, \ldots, y_{n_1}; t)$, periodic in t which satisfy the system

$$\frac{\partial z_j}{\partial t} + \sum_{s=1}^{n_1} \frac{\partial z_j}{\partial y_s}(g_{s1}y_1 + \ldots + g_{sn_1}y_{n_1} + Y_s) = p_{j1}z_1 + \ldots + p_{jp}z_p + Z_j$$

$$(j = 1, \ldots, p)$$

and are equal to zero for $y_1 = \ldots = y_{n_1} = 0$.

Let us transform the system (1.1) into canonic form

$$\xi_s^{\cdot} = \nu_s\xi_s + \Xi_s(\xi_k, \eta_i, t), \qquad \eta_1^{\cdot} = \varkappa_1\eta_1 + H_1(\xi_k, \eta_i, t) \tag{2.4}$$

$$\eta_j^{\cdot} = \varkappa_j\eta_j + \sigma_{j-1}\eta_{j-1} + H_j(\xi_k, \eta_i, t) \qquad (s, k = 1, \ldots, n_1;\ i = 1, \ldots, p;\ j = 2, \ldots, p)$$

We shall consider the system of functions $\eta_j = \eta_j(\xi_1, \ldots, \xi_{n_1}; t)$, satisfying the system

$$\frac{\partial \eta_1}{\partial t} + \sum_{s=1}^{n_1} \frac{\partial \eta_1}{\partial \xi_s}\nu_s\xi_s = \varkappa_1\eta_1 + H_1^{(0)}(\xi_1, \ldots, \xi_{n_1}; t) + \tag{2.5}$$

$$+ H_1^{(1)}(\xi_1, \ldots, \xi_{n_1}; \eta_1, \ldots, \eta_p; t) - \sum_{s=1}^{n_1} \frac{\partial \eta_1}{\partial \xi_s}\Xi_s$$

$$\frac{\partial \eta_j}{\partial t} + \sum_{s=1}^{n_1} \frac{\partial \eta_j}{\partial \xi_s}\nu_s\xi_s = \varkappa_j\eta_j + \sigma_{j-1}\eta_{j-1} + H_j^{(0)}(\xi_1, \ldots, \xi_{n_1}; t) +$$

$$+ H_j^{(1)}(\xi_1, \ldots, \xi_{n_1}; \eta_1, \ldots, \eta_p; t) - \sum_{s=1}^{n_1} \frac{\partial \eta_j}{\partial \xi_s}\Xi_s \qquad (j = 2, \ldots, p)$$

$$H_j^{(0)}(\xi_1, \ldots, \xi_{n_1}; t) = \sum A_j^*(t)\,\xi_1^{k_1}\ldots\xi_{n_1}^{k_{n_1}}$$

$$H_j^{(1)}(\xi_1, \ldots, \xi_{n_1}; \eta_1, \ldots, \eta_p; t) = \sum A_j^{(k_1\ldots k_{n_1} n_1 \ldots n_p)}(t)\,\xi_1^{k_1}\ldots\xi_{n_1}^{k_{n_1}}\eta_1^{n_1}\ldots\eta_p^{n_p}$$

$$(k_1 + \cdots + k_{n_1} \geqslant 2;\quad k_1 + \cdots + k_{n_1} + n_1 + \cdots + n_p \geqslant 2) \tag{2.6}$$

Here the $H^{(1)}_j$ $(j = 1, \ldots, p)$ are equal to zero for $\eta_1 = \ldots = \eta_p = 0$.

We shall write the solutions of the system (2.5) in the form

$$\eta_j = \sum a_j^* \xi_1^{k_1}\ldots\xi_{n_1}^{k_{n_1}} \tag{2.7}$$

where the $a_j^{(k_1\ldots k_{n_1})}(t)$ are periodic functions of t of period 2π, subject to definition. Let us note that as a result of the substitution of η_j in Expression Ξ_s the latter become identically equal to zero.

Substituting the values of η_j into the system (2.5) and identifying the coefficients corresponding to the same powers in $\xi_1^{k_1}\ldots\xi_{n_1}^{k_{n_1}}$, we get linear differential equations for the determination of the coefficients $a_j^{(k_1\ldots k_{n_1})}$:

$$a_1^{\cdot *} + h_1^* a_1^* = A_1^* + P_1^* \tag{2.8}$$

$$a_j^{\cdot *} + h_j^* a_j^* = \sigma_{j-1}a_{j-1}^* + A_j^* + P_j^* \qquad (j = 2, \ldots, p;\ k_1 + \cdots + k_{n_1} = l;\ l = 2, 3 \ldots)$$

where $*$ stands for the superscript $(k_1, \ldots, k_{n_1})$ and $h_j^* = k_1\nu_1 + \ldots + k_{n_1}\nu_{n_1} - \varkappa_j$.

The terms P_j^* represent polynomials of the coefficients $A_j^{(k_1 \ldots k_{n_1} n_1 \ldots n_p)}(t)$ and of the different powers of those a_j^* for which $k_1 + \ldots + k_{n_1} \leqslant l - 1$.

Following the transformations of Liapunov, described in [1] (Sections 35 and 42), we shall prove the convergence of the series (2.7).

Let us determine functions a_j^* for all values of $k_1, \ldots, k_{n_1}$ which satisfy the condition $k_1 + \ldots + k_{n_1} = l$, considering that all the a_j^* for which $k_1 + \ldots + k_{n_1} \leqslant l - 1$ are already known, in the form

$$a_1^* = \frac{e^{-h_1^* t}}{e^{2\pi h_1^*} - 1} \int_t^{t+2\pi} e^{h_1^* t} (A_1^* + P_1^*)\, dt \tag{2.9}$$

$$a_j^* = \frac{e^{-h_j^* t}}{e^{2\pi h_j^*} - 1} \int_t^{t+2\pi} e^{h_j^* t} (\sigma_{j-1} a_{j-1}^* + A_j^* + P_j^*)\, dt \qquad (j = 2, \ldots, p)$$

Let B_j be the largest value of the quantity

$$\frac{1}{|k_1\nu_1 + \cdots + k_{n_1}\nu_{n_1} - \varkappa_j|}$$

for all the values of k_s which satisfy the condition $k_1 + \ldots + k_{n_1} \geqslant 2$, and let the quantities $u_1^*, \ldots, u_p^*$ represent the largest values of the moduli of those $a_1^*, \ldots, a_p^*$ for which $k_1 + \ldots + k_{n_1} \leqslant l - 1$. Let us denote the largest values of the moduli $A_j^*(t)$ by α_j^*, and by ρ_j^* the largest values of the moduli of the expressions P_j^* if in those the values of $A_j^{(k_1 \ldots k_{n_1} n_1 \ldots n_p)}(t)$ are replaced by the largest values of their moduli and if a_j^* is replaced by u_j^* for $k_1 + \ldots + k_{n_1} \leqslant l - 1$.

Expressions (2.9) yield the largest values of the moduli of the a_j^* for which $k_1 + \ldots + k_{n_1} = l$

$$u_1^* = B_1(\alpha_1^* + \rho_1^*), \qquad u_j^* = B_j(|\sigma_{j-1}|\, u_{j-1}^* + \alpha_j^* + \rho_j^*) \quad (j = 2, \ldots, p) \tag{2.10}$$

It is obvious that

$$u_j^{(k_1 \ldots k_{n_1})} \geqslant |a_j^{(k_1 \ldots k_{n_1})}| \qquad (k_1 + \cdots + k_{n_1} = l;\ j = 1, \ldots, p) \tag{2.11}$$

Giving to l the values 2, 3, ..., we determine the largest values of the moduli of all the coefficients entering the series (2.7).

Now let us consider the system of equations

$$\zeta_1 = B_1[F_1^{(0)}(\xi_1, \ldots, \xi_{n_1}) + F_1^{(1)}(\xi_1, \ldots, \xi_{n_1}; \zeta_1, \ldots, \zeta_p)] \tag{2.12}$$

$$\zeta_j = B_j[|\sigma_{j-1}|\, \zeta_{j-1} + F_j^{(0)}(\xi_1, \ldots, \xi_{n_1}) + F_j^{(1)}(\xi_1, \ldots, \xi_{n_1}; \zeta_1, \ldots, \zeta_p)] \quad (j = 2, \ldots, p)$$

where $F_j^{(0)}(\xi_1, \ldots, \xi_{n_1})$; $F_j^{(1)}(\xi_1, \ldots, \xi_{n_1}; \zeta_1, \ldots, \zeta_p)$ $(j = 1, \ldots, p)$ are obtained from $H_j^{(0)}(\xi_1, \ldots, \xi_{n_1}; t)$ and $H_j^{(1)}(\xi_1, \ldots, \xi_{n_1}; \zeta_1, \ldots, \zeta_p; t)$ by substituting for $A_j^*(t)$ and $A_j^{(k_1 \ldots k_{n_1} n_1 \ldots n_p)}(t)$ the largest values of their moduli.

We shall represent the solution of those equations in the form of the series

$$\zeta_j = \sum u_j^* \xi_1^{k_1} \ldots \xi_{n_1}^{k_{n_1}} \qquad (j = 1, \ldots, p), \quad (k_1 + \cdots + k_{n_1} \geqslant 2) \tag{2.13}$$

which are absolutely convergent, at least for sufficiently small values of $|\xi_s|$.

It is simple to show that the coefficients u_j^* are determined from Formulas (2.10).

On the basis of the conditions (2.11) it can be asserted that the series (2.7) are absolutely convergent, at least, for sufficiently small values of $|\xi_s|$. Passing to the original variables y_s, z_j, we get the expressions $z_j = z_j(y_1, \ldots, y_k; t)$ in the form of absolutely convergent series, at least, for sufficiently small values of $|y_s|$.

Going back to the system of equations (2.2), it can be affirmed that the series $u_j(y_1,\ldots,y_{n_1};\ t)$ are absolutely convergent when the conditions of the proved theorem are met.

The system (1.1), after the transformation

$$z_j = \zeta_j + u_j(y_1,\ldots,y_{n_1};\ t)$$

takes the form

$$\begin{aligned} y_s^{\cdot} &= \sum_{k=1}^{n_1} g_{sk}y_k + \sum_{j=1}^{p} P_{sj}^{1}(y_1,\ldots,y_{n_1};t)\,\zeta_j + Y_s^{1}(y_1,\ldots,y_{n_1};\zeta_1,\ldots,\zeta_p;t) \\ \zeta_j^{\cdot} &= \sum_{i=1}^{p} p_{ji}\zeta_i + \sum_{i=1}^{p} Q_{ji}^{1}(y_1,\ldots,y_{n_1};t)\,\zeta_i + Z_j^{1}(y_1,\ldots,y_{n_1};\zeta_1,\ldots,\zeta_p;t) \end{aligned} \tag{2.14}$$

where P_{sj}^{1} and Q_{sj}^{1} are holomorphic functions of $y_1,\ldots,y_n$ which are equal to zero for $y_1 = \ldots = y_{n_1} = 0$ and Y_s^{1} and Z_j^{1} do not contain linear terms in $\zeta_1,\ldots,\zeta_p$.

Let us transform into the canonic form the first group of equations of the system (2.14) by means of a linear transformation with constant real coefficients. We get

$$\begin{aligned} \xi_s^{\cdot} &= -\lambda_s\eta_s + \sum_{j=1}^{p} P_{sj}(\xi_l,\eta_l,r_\mu,t)\,\zeta_j + \Xi_s(\xi_l,\eta_l,r_\mu,\zeta_i,t) \\ \eta_s^{\cdot} &= \lambda_s\xi_s + \sum_{j=1}^{p} S_{sj}(\xi_l,\eta_l,r_\mu,t)\,\zeta_j + H_s(\xi_l,\eta_l,r_\mu,\zeta_i,t) \\ r_k^{\cdot} &= \sum_{j=1}^{p} R_{kj}(\xi_l,\eta_l,r_\mu,t)\,\zeta_j + R_k(\xi_l,\eta_l,r_\mu,\zeta_i,t) \\ \zeta_j^{\cdot} &= \sum_{i=1}^{p} p_{ji}\zeta_i + \sum_{i=1}^{p} Q_{ji}(\xi_l,\eta_l,r_\mu,t)\,\zeta_i + Z_j(\xi_l,\eta_l,r_\mu,\zeta_i,t) \end{aligned} \tag{2.15}$$

$$(s,l = 1,\ldots,q;\ k,\mu = 1,\ldots,m;\ j,i = 1,\ldots,p)$$

Now let us write

$$\xi_s = x_s + \sum_{j=1}^{p}\zeta_j u_{sj}, \qquad \eta_s = y_s + \sum_{j=1}^{p}\zeta_j v_{sj}, \qquad r_k = \rho_k + \sum_{j=1}^{p}\zeta_j w_{kj} \tag{2.16}$$

where x_s, y_s, ρ_k are new variables, and u_{sj}, v_{sj}, w_{kj} are functions of ξ_l, η_l, r_μ and t, satisfying Equations

$$\begin{aligned} \frac{\partial u_{sj}}{\partial t} - \sum_{l=1}^{q}\left(\frac{\partial u_{sj}}{\partial \xi_l}\lambda_l\eta_l - \frac{\partial u_{sj}}{\partial \eta_l}\lambda_l\xi_l\right) &= -\sum_{i=1}^{p} u_{si}p_{ij} - \lambda_s v_{sj} + P_{sj} - \sum_{i=1}^{p} u_{si}Q_{ij} \\ \frac{\partial v_{sj}}{\partial t} - \sum_{l=1}^{q}\left(\frac{\partial v_{sj}}{\partial \xi_l}\lambda_l\eta_l - \frac{\partial v_{sj}}{\partial \eta_l}\lambda_l\xi_l\right) &= -\sum_{i=1}^{p} v_{si}p_{ij} + \lambda_s u_{sj} + S_{sj} - \sum_{i=1}^{p} v_{si}Q_{ij} \\ \frac{\partial w_{kj}}{\partial t} - \sum_{l=1}^{q}\left(\frac{\partial w_{kj}}{\partial \xi_l}\lambda_l\eta_l - \frac{\partial w_{kj}}{\partial \eta_l}\lambda_l\xi_l\right) &= -\sum_{i=1}^{p} w_{ki}p_{ij} + R_{kj} - \sum_{i=1}^{p} w_{ki}Q_{ij} \end{aligned}$$

$$(s = 1,\ldots,q;\ k = 1,\ldots,m;\ j = 1,\ldots,p)$$

This system of equations satisfies all the conditions of the theorem which has just been proved. Consequently, the functions v_{sj}, u_{sj}, w_{kj} are determined in the form of absolutely convergent series with periodic coefficients. After the transformations (2.16), the system of equations (2.15) takes the form

$$x_s^{\cdot} = -\lambda_s y_s + X_s(x_l, y_l, \rho_\mu, \zeta_i, t), \qquad y_s^{\cdot} = \lambda_s x_s + Y_s(x_l, y_l, \rho_\mu, \zeta_i, t)$$

$$\rho_k^{\cdot} = P_k(x_l, y_l, \rho_\mu, \zeta_i, t), \qquad \zeta_j^{\cdot} = \sum_{i=1}^{p} p_{ji}\zeta_i + Z_j^1(x_l, y_l, \rho_\mu, \zeta_i, t) \qquad (2.17)$$

where X_s, Y_s, P_k, Z_j^1 are equal to zero for $\zeta_1 = \ldots = \zeta_p = 0$, and furthermore where the functions X_s, Y_s and P_k do not include linear terms in the variables $\zeta_1, \ldots, \zeta_p$.

Let us take a Liapunov function corresponding to the system (2.17) in the form

$$V = \sum_{s=1}^{q} (x_s^2 + y_s^2) + \sum_{k=1}^{m} \rho_k^2 + W(\zeta_1, \ldots, \zeta_p)$$

where W is a positive definite quadratic form which satisfies Equation

$$\sum_{j=1}^{p} \frac{\partial W}{\partial \zeta_j}(p_{j1}\zeta_1 + \ldots + p_{jp}\zeta_p) = -(\zeta_1^2 + \ldots + \zeta_p^2)$$

The derivative V' can be represented in the form

$$V' = -\sum_{j=1}^{p} \zeta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} \Psi_{ij}\zeta_i\zeta_j$$

where ψ_{ij} are equal to zero for $x_s = y_s = \rho_k = \zeta_j = 0$. Consequently, the unperturbed motion is stable.

Let us consider now the second possible case, which occurs when we have

$$Y_s(y_1, \ldots, y_{n_1}; u_1, \ldots, u_p, t) = \sum Y_{s1}^{(k)}(y_1, \ldots, y_{n_1}; t) \not\equiv 0$$

as a result of the substitution of the variables z_j on the basis of Formulas (2.1).

Let us assume that the lowest of the forms $Y_{s1}^{(k)}$, which is not equal to zero, has an order $h \leq N$. If $h = N$, we can proceed in the following manner. Take the functions v_j equal to the sum of the $N + K$ first terms of the series determining $u_j(y_1, \ldots, y_{n_1}; t)$, taking for K any arbitrary large number. Then the lowest form $Y_{s1}^{(k)}$, which has the order of N, remains without changes and the lowest form $Z_{j1}^{(0)}$ has the order $N + K + 1$. Consequently, in the second case, we can always consider the lowest form $Z_{j1}^{(0)}$ as being lower than the form $Y_{s1}^{(k)}$ by any arbitrary large number K.

Let us show how the system (1.1) is transformed into a new one for which the condition (2) is satisfied. We shall assume that in the system (1.1) all the g_{sk} and p_{ji} are equal to zero with the exception of

$$g_{11} = \nu_1, \ldots, g_{n_1 n_1} = \nu_{n_1}, \qquad p_{11} = \varkappa_1, \ldots, p_{pp} = \varkappa_p$$

$$g_{21} = \sigma_1, \ldots, g_{n_1 n_1 - 1} = \sigma_{n_1 - 1}, \qquad p_{21} = \delta_1, \ldots, p_{pp-1} = \delta_{p-1}$$

We can always bring the system (1.1) into such a form by means of linear substitutions. Let us introduce the change of variables

$$y_s = \eta_s + \sum u_s^* y_1^{k_1} \cdots y_{n_1}^{k_{n_1}} \qquad (1 \leq k_1 + \ldots + k_{n_1} \leq N;\ s = 1, \ldots, n_1) \qquad (2.18)$$

Here the u_s^* are linear forms of $z_1, \ldots, z_p$ having periodic coefficients and satisfying the relations

$$\frac{\partial u_s^*}{\partial t} + \sum_{j=1}^{p} \frac{\partial u_s^*}{\partial z_j} (\varkappa_j z_j + \delta_{j-1} z_{j-1}) = \tag{2.19}$$

$$= -[k_1 \nu_1 + \ldots + (k_s - 1)\nu_s + \ldots + k_{n_1}\nu_{n_1}] u_s^* + \sigma_{s-1} u_{s-1}^* -$$

$$-(k_2+1)\sigma_1 u_s^{(k_1-1\, k_2+1 \ldots k_{n_1})} - \ldots - (k_{n_1}+1)\sigma_{n_1-1} u_s^{(k_1 \ldots k_{n_1-1}-1\, k_{n_1}+1)} + P_s^* + F_s^*$$

The quantities F_s^* $(k_1 + \ldots + k_{n_1} = \delta)$ are polynomials of those u_s^* for which $k_1 + \ldots + k_{n_1} \leqslant \delta - 1$. For $\delta = 1$ all the $F_s^* \equiv 0$.

These equations allow the determination of u_s^* as linear forms of $z_1, \ldots, z_p$ having continuous periodic coefficients of period 2π.

The functions u_s^* $(z_1, \ldots, z_p;\ t)$ can have complex coefficients, which appeared as a result of the linear transformation transforming the coefficients g_{sk} and p_{ij} into zero. If we perform the inverse transformation, then for Expressions

$$u_s^* \quad y_1^{k_1} \ldots y_{n_1}^{k_{n_1}}$$

we get real values. As a result of the transformations (2.1) and (2.18) the system (1.1) takes the form

$$\eta_s^{\cdot} = \sum_{k=1}^{n_1} g_{sk}\eta_k + \sum_{k \geqslant 2}^{N} Y_{s1}^{(k)}(\eta_1, \ldots, \eta_{n_1}; t) + \sum_{k=N+1}^{\infty} Y_{s1}^{(k)}(\eta_1, \ldots, \eta_{n_1}; t) +$$

$$+ \sum_{K=N+1}^{\infty} P_{s1}^*(\zeta_1, \ldots, \zeta_p; t)\, \eta_1^{k_1} \ldots \eta_{n_1}^{k_{n_1}} + H_s(\eta_1, \ldots, \eta_{n_1}; \zeta_1, \ldots, \zeta_p; t)$$

$$\zeta_j^{\cdot} = \sum_{i=1}^{p} p_{ji}\zeta_i + \sum_{k \geqslant N+1}^{\infty} Z_{j1}^{(k)}(\eta_1, \ldots, \eta_{n_1}; t) + E_j(\eta_1, \ldots, \eta_{n_1}; \zeta_1, \ldots, \zeta_p; t)$$

$$(K = k_1 + \ldots + k_{n_1}) \tag{2.20}$$

The transformations of the system (0.1) into the form (2.20) when the a_{sk} are constant, and the X_s do not depend explicitly on time, were presented by the author in [3]. For the system (2.20) to satisfy also the condition (3) it is necessary to transform it to the form in which the functions which play the role of $Y_{s1}^{(k)}$ $(k \leqslant N)$, have constant coefficients.

It is sufficient to show the possibility of such a transformation for the "shortened" system

$$\eta_s^{\cdot} = \sum_{k=1}^{n_1} g_{sk}\eta_k + \sum_{k \geqslant 2}^{N} Y_{s1}^{(k)}(\eta_1, \ldots, \eta_{n_1}; t), \qquad Y_{s1}^{(k)} = \sum A_s^*(t)\, \eta_1^{k_1} \ldots \eta_{n_1}^{k_{n_1}} \tag{2.21}$$

3. Let us assume at first that the characteristic equation of the system (2.21) has m roots equal to zero to which there correspond m groups of solutions and q pairs of pure imaginary roots $\pm i\lambda_s$ satisfying the condition

$$\sum_{s=1}^{q} m_s\lambda_s \neq E \quad \text{for} \quad 2 \leqslant \sum_{s=1}^{q} |m_s| \leqslant N \tag{3.1}$$

where the m_s and E are integers, including zero.

In this case the system (2.21) can be transformed into the form

$$x_s^{\cdot} = -\lambda_s y_s + X_s(x_i, y_i, \xi_r, t), \qquad y_s^{\cdot} = \lambda_s x_s + Y_s(x_i, y_i, \xi_r, t)$$

$$\xi_j^{\cdot} = \Xi_j(x_i, y_i, \xi_r, t) \qquad (s, i = 1, \ldots, q;\ j, r = 1, \ldots, m) \tag{3.2}$$

Assuming $z_s = x_s + iy_s$, $\bar{z}_s = x_s - iy_s$, we get

$$z_s^{\cdot} = i\lambda_s z_s + Z_s(z_i, \bar{z}_i, \xi_r, t), \qquad \bar{z}_s^{\cdot} = -i\lambda_s \bar{z}_s + \bar{Z}_s(z_i, \bar{z}_i, \xi_r, t)$$

$$\xi_j^{\cdot} = P_j(z_i, \bar{z}_i, \xi_r, t) \qquad (s, i = 1, \ldots, q;\ j, r = 1, \ldots, m) \tag{3.3}$$

Here

$$Z_s = \sum_{l \geqslant 2}^{N} Z_s^{(l)}(z_i, \bar{z}_i, \xi_r, t), \quad \bar{Z}_s = \sum_{l \geqslant 2}^{N} \bar{Z}_s^{(l)}(z_i, \bar{z}_i, \xi_r, t)$$

$$P_j = \sum_{l \geqslant 2}^{N} P_j^{(l)}(z_i, \bar{z}_i, \xi_r, t)$$

and $Z_s^{(l)}$, $\bar{Z}_s^{(l)}$ and $P_j^{(l)}$ are forms of the lth order in z_1, $\bar{z}_1$ and ξ_r, which can be represented in the form

$$Z_s^{(l)} = \sum A_s^*(t)\, z_1^{k_1} \ldots z_q^{k_q} \bar{z}_1^{m_1} \ldots \bar{z}_q^{m_q} \xi_1^{\delta_1} \ldots \xi_m^{\delta_m}$$

$$\bar{Z}_s^{(l)} = \sum \bar{A}_s^*(t)\, \bar{z}_1^{k_1} \ldots \bar{z}_q^{k_q} z_1^{m_1} \ldots z_q^{m_q} \xi_1^{\delta_1} \ldots \xi_m^{\delta_m} \tag{3.4}$$

$$P_j^{(l)} = \sum B_j^*(t)\, z_1^{k_1} \ldots z_q^{k_q} \bar{z}_1^{m_1} \ldots \bar{z}_q^{m_q} \xi_1^{\delta_1} \ldots \xi_m^{\delta_m}, \qquad A_s^*(t) = A_s^*(t + 2\pi)$$

Here and further on in this Section, the superscript * replaces the index $(k_1 \ldots k_q m_1 \ldots \ldots m_q \delta_1 \ldots \delta_m)$.

Let us rewrite the system (3.3) using variables ζ_s, $\bar{\zeta}_s$ and η_j, setting

$$z_s = \zeta_s + u_s(z_i, \bar{z}_i, \xi_r, t), \quad \bar{z}_s = \bar{\zeta}_s + \bar{u}_s(z_i, \bar{z}_i, \xi_r, t), \quad \xi_j = \eta_j + v_j(z_i, \bar{z}_i, \xi_r, t)$$

and considering u_s, $\bar{u}_s$, v_j as being periodic functions of t subject to definition. We shall represent these functions in the form

$$u_s = \sum u_s^*(t)\, z_1^{k_1} \ldots z_q^{k_q} \bar{z}_1^{m_1} \ldots \bar{z}_q^{m_q} \xi_1^{\delta_1} \ldots \xi_m^{\delta_m} \qquad (s = 1, \ldots, q;\ j = 1, \ldots, m)$$

$$v_j = \sum v_j^*(t)\, z_1^{k_1} \ldots z_q^{k_q} \bar{z}_1^{m_1} \ldots \bar{z}_q^{m_q} \xi_1^{\delta_1} \ldots \xi_m^{\delta_m}$$

The transformed system takes the form

$$\zeta_s^{\cdot} = i\lambda_s \zeta_s + \sum_{l \geqslant 2}^{\infty} Z_{s1}^{(l)}, \qquad \bar{\zeta}_s^{\cdot} = -i\lambda_s \bar{\zeta}_s + \sum_{l \geqslant 2}^{\infty} \bar{Z}_{s1}^{(l)}, \qquad \eta_j^{\cdot} = \sum_{l \geqslant 2}^{\infty} H_j^{(l)} \tag{3.5}$$

The functions $Z_{s1}^{(l)}$ and $H_j^{(l)}$ can be represented as

$$Z_{s1}^{(l)} = \sum a_s^*(t)\, \zeta_1^{k_1} \ldots \zeta_q^{k_q} \bar{\zeta}_1^{m_1} \ldots \bar{\zeta}_q^{m_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} \qquad (s = 1, \ldots, q) \tag{3.6}$$

$$H_j^{(l)} = \sum b_j^*(t)\, \zeta_1^{k_1} \ldots \zeta_q^{k_q} \bar{\zeta}_1^{m_1} \ldots \bar{\zeta}_q^{m_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} \qquad (j = 1, \ldots, m)$$

The coefficients $a_s^*(t)$ and $b_j^*(t)$ for

$$k_1 + \ldots + k_q + m_1 + \ldots + m_q + \delta_1 + \ldots + \delta_m = k$$

have the form (3.7)

$$a_s^* = -\frac{du_s^*}{dt} - i\,[(k_1 - m_1)\lambda_1 + \ldots + (k_s - m_s - 1)\lambda_s + \ldots + (k_q - m_q)\lambda_q]\, u_s^* + A_s^*(t) + F_s^*(u_s^{(l)}, \bar{u}_s^{(l)}, v_j^{(l)}, t)$$

$$b_j^* = -\frac{dv_j^*}{dt} - i\,[(k_1 - m_1)\lambda_1 + \ldots + (k_s - m_s)\lambda_s + \ldots + (k_q - m_q)\lambda_q]\, v_j^* + B_j^*(t) + \Phi_j^*(u_s^{(l)}, \bar{u}_s^{(l)}, v_j^{(l)}, t)$$

where the F_s^* and Φ_j^* are known functions of t and those $u_s^{(l)}$, $v_j^{(l)}$, $\bar{u}_s^{(l)}$, for which $l \leq k-1$. For $k = 2$ all the F_s^* and Φ_j^* are identically equal to zero. The coefficients $\bar{a}_s^*$ are determined by analogous relations.

From (3.7) there follows that for different values of the functions u_s^*, $\bar{u}_s^*$ and v_j^* we get different values of the coefficients a_s^*, $\bar{a}_s$ and b_j^*. We shall determine u_s^* $\bar{u}_s^*$ and v_j^* in such a way that a_s^*, $\bar{a}_s^*$ and b_j^* are equal to zero or to constant quantities.

Let us assume that all the functions u_s^* and v_j^* for which

$$k_1 + \ldots + k_q + m_1 + \ldots + m_q + \delta_1 + \ldots + \delta_m = k - 1$$

are determined from the conditions $a_s^* = 0$, $b_j^* = 0$, or $a_s^* = \text{const}$, $b_j^* = \text{const}$.

We shall determine u_s^* and v_j^* for

$$k_1 + \ldots + k_q + m_1 + \ldots + m_q + \delta_1 + \ldots + \delta_m = k$$

Let us consider the set of the numbers k_s and m_s which satisfy the condition

$$d = (k_1 - m_1)\lambda_1 + \ldots + (k_s - m_s - 1)\lambda_s + \ldots + (k_q - m_q)\lambda_q \neq 0$$
$$(k_1 + \ldots + k_q + m_1 + \ldots + m_q + \delta_1 + \ldots + \delta_m = k)$$

It is evident that for such values of k_s and m_s we can determine the functions u_s^* for any arbitrary value of a_s^*. We shall find these functions for the conditions $a_s^* = 0$.

Let us note that on the basis of (3.1), the equality $d = 0$ is possible only for

$$k_1 = m_1, \ldots, \ k_s = m_s + 1, \ldots, \ k_q = m_q \tag{3.8}$$

We shall determine the coefficients a_s^* corresponding to the index $(k_1 \ldots k_s \ldots$ $\ldots k_q \; k_1 \ldots k_s - 1 \ldots k_q \delta_1 \ldots \delta_m)$, by the relations

$$a_s^* = \frac{1}{2\pi}\int_0^{2\pi} (A_s^* + F_s^*)\, dt$$

Then the periodic functions u_s^* having the same index are determined from Equations

$$\frac{du_s^*}{dt} = -a_s^* + A_s^* + F_s^* \qquad (s = 1, \ldots, q) \tag{3.9}$$

Let us determine the functions v_j in the following manner. Let us seek the periodic functions v_j^* with the condition $b_j^* = 0$ for the numbers k_s and m_s satisfying the condition

$$d_1 = (k_1 - m_1)\lambda_1 + \ldots + (k_s - m_s)\lambda_s + \ldots + (k_q - m_q)\lambda_q \neq 0$$
$$(k_1 + \ldots + k_q + m_1 + \ldots + m_q + \delta_1 + \ldots + \delta_m = k)$$

Noting that the equality $d_1 = 0$ is possible only for

$$k_1 = m_1, \ldots, \ k_q = m_q \tag{3.10}$$

We shall determine the b_j^* corresponding to the index $(k_1 \ldots k_q k_1 \ldots k_q \delta_1 \ldots \delta_m)$, by Equations

$$b_j^* = \frac{1}{2\pi}\int_0^{2\pi} (B_j^* + \Phi_j^*)\, dt$$

and the periodic functions v_j from Equations

$$\frac{dv_j^*}{dt} = B_j^* + \Phi_j^* - b_j^* \qquad (j = 1, \ldots, m)$$

Consequently we can assert that in Expressions (3.6) the forms $Z_{s1}^{(l)}$ will contain

contain only those terms for which the powers of k_s and m_s satisfy the condition (3.8), and the forms $H_j^{(l)}$ will retain only those powers which satisfy the condition (3.10).

Assigning to the term k the values 2, 3, ..., N, we shall determine all the a_s^* and b_j^* for which $k_1 + \ldots + k_q + m_1 + \ldots + m_q + \delta_1 + \ldots + \delta_m \leqslant N$.

As a result of the transformation the system of equations (3.5) takes on the form

$$\zeta_s^{\cdot} = i\lambda_s\zeta_s + \zeta_s \sum a_s^* (\zeta_1\bar{\zeta}_1)^{k_1} \ldots (\zeta_q\bar{\zeta}_q)^{k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} + Z_s^{(N+1)} (\zeta_i, \bar{\zeta}_i, \eta_r, t)$$

$$\bar{\zeta}_s^{\cdot} = -i\lambda_s\bar{\zeta}_s + \bar{\zeta}_s \sum a_s^* (\zeta_1\bar{\zeta}_1)^{k_1} \ldots (\zeta_q\bar{\zeta}_q)^{k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} + \bar{Z}_s^{(N+1)} (\zeta_i, \bar{\zeta}_i, \eta_r, t)$$

$$\eta_j^{\cdot} = \sum b_j^* (\zeta_1\bar{\zeta}_1)^{k_1} \ldots (\zeta_q\bar{\zeta}_q)^{k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} + H_j^{(N+1)} (\zeta_i, \bar{\zeta}_i, \eta_r, t) \quad (3.11)$$

$$(s, i = 1, \ldots, q;\ j, r = 1, \ldots, m;\ 2 \leqslant 2k_1 + \ldots + 2k_q + \delta_1 + \ldots + \delta_m \leqslant N)$$

where $Z_s^{(N+1)}$, $\bar{Z}_s^{(N+1)}$ and $H_j^{(N+1)}$ do not contain terms of order lower than $(N+1)$.

Investigating the canonic systems in the case of irrational λ_s and assuming $m = 0$, Birkhof [4] has obtained an analogous system. Assuming

$$\zeta_s = r_s e^{i\theta_s}, \qquad a_s^* = \alpha_s^* + i\beta_s^*$$

we get

$$r_s^{\cdot} = r_s \sum \alpha_s^* r_1^{2k_1} \ldots r_q^{2k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} + R_s^{(N+1)}$$

$$r_s\theta_s^{\cdot} = \lambda_s r_s + r_s \sum \beta_s^* r_1^{2k_1} \ldots r_q^{2k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} + F_s^{(N+1)} \quad (3.12)$$

$$\eta_j^{\cdot} = \sum b_j^* r_1^{2k_1} \ldots r_q^{2k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m} + H_j^{(N+1)}.$$

If the question of stability is solved by the terms of the Nth order independently of the terms of higher order, the problem reduces to the investigation of a system of equations of the $(m+q)$ order of the form

$$r_s^{\cdot} = r_s \sum \alpha_s^* r_1^{2k_1} \ldots r_q^{2k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m}, \quad \eta_j^{\cdot} = \sum b_j^* r_1^{2k_1} \ldots r_q^{2k_q} \eta_1^{\delta_1} \ldots \eta_m^{\delta_m}$$

$$(2 \leqslant 2k_1 + \ldots + 2k_q + \delta_1 + \ldots + \delta_m \leqslant N) \quad (3.13)$$

On the basis of what has been presented, there follows that the investigation of the system of equations (0.1) satisfying the condition (3.1) and the characteristic equation of which has p roots of moduli smaller than unity, m roots equal to unity and $2q$ roots of moduli equal to unity (roots of the form $\nu_s = e^{\pm 2\pi i\lambda_s}$), can be reduced to the investigation of the integrals of a system of equations with constant coefficients characterized by $(m+q)$ zero roots with $(m+q)$ groups of solutions.

4. Before we pass to the general case, let us consider the system (2.21) assuming that its characteristic equation has, at least, one pair of pure imaginary roots $\pm i\lambda_1$ of multiplicity r. Then, by means of a linear substitution with constant real coefficients, we can transform this system into the form

$$\eta_s^{\cdot} = \sum_{k=1}^{n_1-2r} b_{sk}\eta_k + H_s(\eta_i, x_\nu, y_\nu, t)$$

$$x_1^{\cdot} = -\lambda_1 y_1 + X_1(\eta_i, x_\nu, y_\nu, t), \qquad y_1^{\cdot} = \lambda_1 x_1 + Y_1(\eta_i, x_\nu, y_\nu, t) \quad (4.1)$$

$$x_j^{\cdot} = -\lambda_1 y_j + \sigma_{j-1} x_{j-1} + X_j(\eta_i, x_\nu, y_\nu, t)$$

$$y_j^{\cdot} = \lambda_1 x_j + \sigma_{j-1} y_{j-1} + Y_j(\eta_i, x_\nu, y_\nu, t)$$

$$(s, i = 1, \ldots, n_1 - 2r;\ j = 2, \ldots, r;\ \nu = 1, \ldots, r)$$

where x_j, y_j are linear terms in η_s ($s = 1, \ldots, n_1$). Equation $|b_{sk} - \delta_{sk}\nu| = 0$ has m roots equal to zero and $2(q-r)$ pure imaginary roots.

Let us notice that all the quantities σ_{j-1}, if different from zero, can be considered as being equal to any arbitrary number. We shall consider them equal to λ_1. Assuming

$$
\begin{aligned}
x_1 &= \xi_1 \cos \lambda_1 t + \zeta_1 \sin \lambda_1 t, \ y_1 = \xi_1 \sin \lambda_1 t - \zeta_1 \cos \lambda_1 t \\
x_j &= \xi_j \cos\lambda_1 t + \zeta_j \sin\lambda_1 t + \xi_{j-1} \cos \lambda_1 t + \zeta_{j-1} \sin \lambda_1 t \qquad (4.2) \\
y_j &= \xi_j \sin \lambda_1 t - \zeta_j \cos\lambda_1 t + \xi_{j-1} \sin\lambda_1 t - \zeta_{j-1} \cos \lambda_1 t
\end{aligned}
$$

we have

$$
\eta_s' = \sum_{k=1}^{n_1-2r} b_{sk}\eta_k + H_{s1}(\eta_i, \xi_\nu, \zeta_\nu, t)
$$

$$
\xi_1' = \Xi_1(\eta_i, \xi_\nu, \zeta_\nu, t), \qquad \zeta_1' = Z_1(\eta_i, \xi_\nu, \zeta_\nu, t) \qquad (4.3)
$$

$$
\xi_j' = \lambda_1 \xi_{j-1} + \Xi_j(\eta_i, \xi_\nu, \zeta_\nu, t), \qquad \zeta_j' = \lambda_1 \zeta_{j-1} + Z_j(\eta_i, \xi_\nu, \zeta_\nu, t)
$$

$$
(s, i = 1, \ldots, n_1 - 2r; \ j = 2, \ldots, r; \ \nu = 1, \ldots, r)
$$

If a few groups of solutions correspond to each multiple root $\pm i\lambda_1$, then for each such group the transformations described by Formulas (4.2) are carried out completely analogously.

Let us note that the characteristic equation of the system (4.3) has $m + 2r$ roots equal to zero and $2(q-r)$ pure imaginary roots.

If one group of solutions corresponds to the pure imaginary roots $\pm i\lambda_1$, i.e. all the $\sigma_{j-1} \neq 0$, then two groups of solutions will correspond to the complementary zero roots. If, however, k group of solutions will correspond to those roots, then $2r$ zero roots will have $2k$ groups of solutions.

Let us notice also that when λ_1 is equal to an integer, the functions H_{s1}, Ξ_j, Z_j are periodic functions of t with a period 2π.

If $\lambda_1 = \alpha_1/\beta_1$ (α_1, β_1 are integers), then by substituting $t = \beta_1 \tau$, we give the system a form for which λ_1 is equal to an integer.

In the case of irrational λ_1, the problem is somewhat more complicated. The expressions H_{s1}, Ξ_j, Z_j, are not periodic functions of t anymore. Let us represent one of them in the form

$$
H_{s1} = \sum H_s^{(0)}(t)\, \eta_1^{\gamma_1} \ldots \eta_p^{\gamma_p} \xi_1^{m_1} \ldots \xi_r^{m_r} \zeta_1^{\delta_1} \ldots \zeta_r^{\delta_r} \qquad (4.4)
$$

$$
(p = n_1 - 2r; \ s = 1, \ldots, p), \ (2 \leqslant \gamma_1 + \ldots + \gamma_p + m_1 + \ldots + m_r + \delta_1 + \ldots + \delta_r \leqslant N)
$$

where (0) replaces the index $(\gamma_1 \ldots \gamma_p m_1 \ldots m_r \delta_1 \ldots \delta_r)$.

It is easy to find that the functions $H_s^{(0)}(t)$ are linear expressions of the coefficients $A_s^{(k_1 \ldots k_{n_1})}(t)$, appearing in the system (2.21), multiplied by $\sin \epsilon_1 \lambda_1 t$ and $\cos \epsilon_1 \lambda_1 t$. The numbers ϵ_1 appearing in forms of the l th order can take values from 1 to l.

There follows that the functions $H_s^{(0)}(t)$ for which

$$
\gamma_1 + \ldots + \gamma_p + m_1 + \ldots + m_r + \delta_1 + \ldots + \delta_r = l
$$

can be represented in the form

$$
H_s^{(0)}(t) = \sum_{\epsilon_1} A_{s1}^{(0)}(t)\, e^{i\epsilon_1\lambda_1 t} \qquad (4.5)
$$

where $A_{s1}^{(0)}(t)$ are periodic functions of t having a period 2π. When λ_1 is irrational, the $H_s^{(0)}(t)$ are almost periodic functions of t.

The coefficients of the expansions of Ξ_j and Z_j appearing in the system (4.3) have a similar structure.

A similar transformation can be made for any pair of simple or multiple pure imaginary roots $\pm i\lambda_s$. Consequently, in the general case the system (2.21) can always be transformed into a new system, the characteristic equation of which has all its roots equal to zero. This system can be represented in the form

$$x_1' = X_1(x_1, \ldots, x_n; t), \qquad x_s' = \gamma_{s-1}x_{s-1} + X_s(x_1, \ldots, x_n; t) \qquad (s = 2, \ldots, n;\; n = n_1) \tag{4.6}$$

where

$$X_s = \sum_{l\geqslant 2}^{N} X_s^{(l)}(x_1, \ldots, x_n; t), \qquad X_s^{(l)} = \sum B_s^*(t)\, x_1^{k_1} \ldots x_n^{k_n} \qquad (k_1 + \ldots + k_n = l),\; (s = 1, \ldots, n)$$

The symbol $*$ stands for the index $(k_1, \ldots, k_n)$.

The function $B_s^*(t)$ have the following structure:

$$B_s^*(t) = \sum_{\varepsilon_j}\sum_{k_s} A^{\bullet}_{s**}(t) \exp i\,(\varepsilon_1\lambda_1 + \varepsilon_2\lambda_2 + \ldots + \varepsilon_\mu\lambda_\mu)\, t \tag{4.7}$$

The symbol $**$ stands for the index $(\varepsilon_1 \ldots \varepsilon_\mu)$. The quantity μ determines the number of irrational pure imaginary roots. The summation with respect to k_s extends to all the positive integers k_s which satisfy the equality $k_1 + \ldots + k_n = \ell$, and the summation with respect to ε_j to all the positive and negative numbers ε_j $(j = 1, \ldots, \mu)$ which satisfy the condition $\sum |\varepsilon_j| \leq \ell$.

The functions $A^{\bullet}_{s**}$ are representative linear forms with constant complex coefficients of $A_s^*(t)$; they are periodic in t with a period 2π.

The functions $B_s^*(t)$ are real, almost periodic functions of t for real values of t.

The new variables $x_1, \ldots, x_n$ are real functions of the real variable t.

We should point out, that the problem of stability with respect to the variables η_s of the original system (2.21) and x_s are equivalent.

5. Let us prove now that for cases which are not essentially singular, we can always reduce the problem of the stability of periodic oscillations, characterized by the system (2.21) to the problem of the stability of equilibrium.

Let us consider the system (4.6) assuming that somehow it was possible to transform it into a form for which all the forms $X_s^{(l)}$ are independent of time t for $\ell \leq k-1$.

We shall transform this system, writing

$$x_s = y_s + \sum u_s^*(t)\, x_1^{k_1} \ldots x_n^{k_n} \qquad (k_1 + \ldots + k_n = k;\; s = 1, \ldots, n) \tag{5.1}$$

where

$$u_s^*(t) = \sum_{\varepsilon_j}\sum_{k_s} u^{\bullet}_{s**}(t) \exp i\,(\varepsilon_1\lambda_1 + \ldots + \varepsilon_\mu\lambda_\mu)\, t \tag{5.2}$$

The indices $*$ and $**$ have the same meaning as in Formula (4.7). From (5.1) we get

$$x_s = y_s + \sum v_s^*(t)\, y_1^{k_1} \ldots y_n^{k_n} \qquad (k_1 + \ldots + k_n \geqslant k) \tag{5.3}$$

The functions v_s^* are equal to u_s^* for $k_1 + \ldots + k_n = k$: for the values $k_1 + \ldots + k_n > k = \ell$ these functions are polynomials in those u_s^* for which $k_1 + \ldots + k_n \leq \ell - 1$.

Taking (4.6) into consideration, we get from (5.1) and (5.3)

$$y_1^{\cdot} = \sum_{l\geq 2}^{\infty} Y_1^{(l)}(y_1,\ldots,y_n;t), \qquad y_s^{\cdot} = \gamma_{s-1}y_{s-1} + \sum_{l\geq 2}^{\infty} Y_s^{(l)}(y_1,\ldots,y_n;t)$$

$$(s=2,\ldots,n;\ l=2,3,\ldots) \tag{5.4}$$

In this system, when $l \leqslant k-1$ $(s=1,\ldots,n)$ the forms $Y_s^{(l)}(y_1,\ldots,y_n;t)$ are equal to $X_s^{(l)}(x_1,\ldots,x_n)$ where x_s is replaced by y_s, and the forms $Y_s^{(k)}(y_1,\ldots,y_n;t)$ become

$$Y_s^{(k)}(y_1,\ldots,y_n;t) = \sum \alpha_s^*(t)\, y_1^{k_1}\ldots y_n^{k_n} \qquad (k_1+\ldots+k_n=k;\ s=1,\ldots,n) \tag{5.5}$$

The functions $a_s^*(t)$ are determined by the equalities

$$\alpha_s^*(t) = -\frac{du_s^*}{dt} - (k_2+1)\gamma_1 u_s^{(k_1-1k_2+1k_3\ldots k_n)} - (k_3+1)\gamma_2 u_s^{(k_1k_2-1k_3+1\ldots k_n)} -$$
$$- \ldots - (k_n+1)\gamma_{n-1}u_s^{(k_1\ldots k_{n-1}-1k_n+1)} + \gamma_{s-1}u_{s-1}^{*} + B_s^*(t) \tag{5.6}$$
$$(s=1,\ldots,n;\ k_1+\ldots+k_n=k)$$

Different values are obtained from the functions $a_s^*(t)$ by giving different values to the functions $u_s^*(t)$. The functions $u_s^*(t)$ are determined so that the $a_s^{**}(t)$ become equal to zero or to constants.

It is evident that the functions $u_{s**}^*(t)$ can be determined from Equations

$$-\frac{du_{s**}^*}{dt} - i(\varepsilon_1\lambda_1 + \ldots + \varepsilon_\mu\lambda_\mu)\, u_{s**}^* + B_{s**}^* + L_{s**}^* = \alpha_{s**}^* \tag{5.7}$$

where the L_{s**}^* are periodic functions of t, with a period 2π, which are linear forms of the already found $u_{s**}^{(k_1\ldots k_{j-1}-1k_{j+1}\ldots k_n)}$.

If the numbers $\varepsilon_1,\ldots,\varepsilon_\mu$ are such that $\varepsilon_1\lambda_1+\ldots+\varepsilon_\mu\lambda_\mu \neq 0$, then the periodic function $u_{s**}^*(t)$ can be determined by assuming that $a_{s**}^* = 0$.

If, however, for some of the numbers $\varepsilon_1,\ldots,\varepsilon_\mu$ the relation $\varepsilon_1\lambda_1+\ldots+\varepsilon_\mu\lambda_\mu = 0$ is satisfied, then the $\alpha_{s\varepsilon_1\ldots\varepsilon_\mu}^*$, corresponding to those values of $\varepsilon_1,\ldots,\varepsilon_\mu$, must be determined from the equalities

$$\alpha_{s\varepsilon_1\ldots\varepsilon_\mu}^* = \frac{1}{2\pi}\int_0^{2\pi} (B_{s\varepsilon_1\ldots\varepsilon_\mu}^* + L_{s\varepsilon_1\ldots\varepsilon_\mu}^*)\, dt \qquad (s=1,\ldots,n;\ k_1+\ldots+k_n=k) \tag{5.8}$$

Then Equations (5.7) determine $u_{s\varepsilon_1\ldots\varepsilon_\mu}^*(t)$, in the form of periodic functions of period 2π. Consequently, as a result of the transformation (5.1), for the values of the functions u_s^* found in the system of equations (5.4), all the forms $Y_s^{(k)}$ have constant coefficients. It is evident that the structure of the forms $Y_s^{(l)}$ for $l > k$ is the previous one, i.e. (4.7).

It is also evident, that the problems of stability with respect to the variables $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are equivalent.

By giving to the number k the values $2, 3, \ldots, N$, we transform the system (4.6) into a new system of the form

$$z_1^{\cdot} = \sum a_1^* z_1^{k_1}\ldots z_n^{k_n} + \sum_{l=N+1}^{\infty} Z_1^{(l)}(z_1,\ldots,z_n;t)$$

$$z_s^{\cdot} = \gamma_{s-1} z_{s-1} + \sum a_s^* z_1^{k_1} \ldots z_n^{k_n} + \sum_{l=N+1}^{\infty} Z_s^{(l)} (z_1, \ldots, z_n; t) \qquad (5.9)$$

$$(s = 2, \ldots, n;\ 2 \leqslant k_1 + \ldots + k_n \leqslant N)$$

The points exposed in Sections 3 to 5 allow us to formulate the following theorem.

Theorem 5.1 . If the system of equations (0.1) is such, that its characteristic equation has m roots equal to unity, q pairs of roots of moduli equal to unity, and p roots with moduli smaller than unity, the problem of the stability of periodic motions, characterized by that system, for cases not essentially singular, can always be reduced to the problem of the stability of an equilibrium.

If the roots $\nu_s = e^{\pm 2\pi i \lambda_s}$, the moduli of which are equal to unity, satisfy the condition (3.1), then the problem of the stability of equilibrium, to the investigation of which was reduced the problem of the stability of periodic oscillations, is characterized by $(m+q)$ roots equal to zero .

6 . As an example, let us consider the problem of stability when the characteristic equation has two roots of moduli equal to unity, one root equal to unity and n roots with moduli smaller than unity .

This problem reduces to the investigation of a system of equations of the form

$$x^{\cdot} = -\lambda y + X(x, y, z, x_k, t),\ y^{\cdot} = \lambda x + Y(x, y, z, x_k, t) \qquad (s = 1, \ldots, n)$$

$$z^{\cdot} = Z(x, y, z, x_k, t), \quad x_s^{\cdot} = \sum_{k=1}^{n} p_{sk} x_k + X_s(x, y, z, x_k, t) \qquad (6.1)$$

Let us transform the system (6.1) by writing

$$x_s = y_s + v_s(x, y, z, t) \qquad (s = 1, \ldots, n) \qquad (6.2)$$

considering v_s as being polynomials, representing the N first terms of the series u_s, satisfying the system of equations

$$\frac{\partial u_s}{\partial t} + \frac{\partial u_s}{\partial x}[-\lambda y + X(x, y, z, u_k, t)] + \frac{\partial u_s}{\partial y}[\lambda x + Y(x, y, z, u_k, t)] + \qquad (6.3)$$

$$+ \frac{\partial u_s}{\partial z} Z(x, y, z, u_k, t) = \sum_{k=1}^{n} p_{sk} u_k + X_s(x, y, z, u_k, t) \qquad (s, k = 1, \ldots, n)$$

If

$$X[x, y, z, u_k(x, y, z, t), t] = Y[x, y, z, u_k(x, y, z, t), t] = Z[x, y, z, u_k(x, y, z, t), t] \equiv 0 \qquad (6.4)$$

then the system (6.3) satisfies all the conditions of the theorem proven in Section 2 . Consequently, the series $u_s(x, y, z, t)$ are absolutely convergent, at least, for sufficiently small values of $|x|$, $|y|$, $|z|$.

Then as the result of the substitution

$$x_s = y_s + u_s(x, y, z, t) \qquad (s = 1, \ldots, n)$$

we have

$$x^{\cdot} = -\lambda y + X_1(x, y, z, y_k, t), \quad y^{\cdot} = \lambda x + Y_1(x, y, z, y_k, t)$$

$$z^{\cdot} = Z_1(x, y, z, y_k, t), \qquad y_s^{\cdot} = \sum_{k=1}^{n} p_{sk} y_k + Y_{s1}(x, y, z, y_k, t) \qquad (s = 1, \ldots, n) \quad (6.5)$$

The functions X_1, Y_1, Z_1, Y_{s1} are identically equal to zero for $y_1 + \ldots = y_n = 0$.

Basing ourselves on the conclusions of Section 2, we can assert that the unperturbed motion is stable .

Note 6.1. The system (6.5) has the particular solution

$$x = c_1 \cos \lambda (t - t_0), \quad y = c_1 \sin\lambda (t - t_0), \quad z = c_2, \; y_1 = \ldots = y_n = 0$$

Consequently, the system (6.1) has an almost periodic solution of the form

$$x_s = u_s [c_1 \cos\lambda (t - t_0), c_1 \sin \lambda (t - t_0), c_2, t]$$

$$x = c_1 \cos \lambda (t - t_0), \quad y = c_1 \sin \lambda (t - t_0), \quad z = c_2 \tag{6.6}$$

which exists, at least, for sufficiently small values of $|c_1|$ and $|c_2|$.

7. Let us assume now that the identities (6.4) do not hold. Then in the case of an irrational λ we can transform the system (6.1) into the form

$$r^{\cdot} = r [R^{(m)} (r^2, z) + \cdots + R^{(N)} (r^2, z)] + R^{(N+1)} (r, z, \theta, t) + R (r, z, \theta, y_k, t)$$

$$z^{\cdot} = Z^{(m_1)} (r^2, z) + \cdots + Z^{(N)} (r^2, z) + Z^{(N+1)} (r, z, \theta, t) + Z (r, z, \theta, y_k, t)$$

$$y_s^{\cdot} = \sum_{k=1}^{n} p_{sk} y_k + Y_s^{(N+1)} (r, z, \theta, t) + Y_s (r, z, \theta, y_k, t)$$

$$r \frac{d\theta}{dt} = \lambda r + F (r, z, \theta, y_k, t) \tag{7.1}$$

where $R^{(l)}$, $Z^{(l)}$ are forms of lth order in $r^2, z \; (l \leqslant N)$, $R^{(N+1)}, Z^{(N+1)}, Y_s^{(N+1)}$ are the ensemble of the terms of order higher than the Nth, and R, Z, Y_s become equal to zero for $y_1 = \ldots = y_n = 0$. The functions R and Z either do not have linear terms in $y_1, \ldots, y_n$ or contain them in products with $r^{k_1} z^{k_2} \; (k_1 + k_2 \geqslant N)$.

If the question of stability with respect to the variables r, z is solved by means of forms $R^{(l)}$ and $Z^{(l)}$ with the condition $l \leq N$, and independently from the forms of higher order, then when investigating the stability of the integrals of the system (7.1) it is sufficient to consider the second order system

$$r^{\cdot} = r [R^{(m)} (r^2, z) + \cdots + R^{(N)} (r^2, z)], \qquad z^{\cdot} = Z^{(m_1)} (r^2, z) + \cdots + Z^{(N)} (r^2, z) \tag{7.2}$$

This statement is proved in [3]. Assuming $r^2 = \rho$ we have

$$\rho^{\cdot} = 2\rho [R^{(m)} (\rho, z) + \cdots + R^{(N)} (\rho, z)], \qquad z^{\cdot} = Z^{(m_1)} (\rho, z) + \cdots + Z^{(N)} (\rho, z) \tag{7.3}$$

When investigating the stability we must consider the variable ρ as being a positive quantity. Such a system is considered in [5] where a very simple case is investigated in which the system (7.2) has the form

$$\rho^{\cdot} = R^{(m)} (\rho, z) + R^{(m+1)} (\rho, z) + \cdots, \qquad z^{\cdot} = Z^{(m)} (\rho, z) + Z^{(m+1)} (\rho, z) + \cdots \tag{7.4}$$

and the question of the stability is solved by forms of the mth order, independently from the forms of higher order. The results of these investigations are also given in [6] and [3].

The problem of the stability of the integrals of the system (7.4) for the solution of which it is indispensable to consider forms of order higher than the mth, is considered in [7].

In the general case, the system of equations (7.3) can be represented in the form

$$\rho^{\cdot} = \rho \, (a^{(2,0)}\rho + a^{(1,1)}z + a^{(3,0)}\rho^2 + a^{(2,1)}\rho z + a^{(1,2)}z^2 + \sum_{k_1+k_2=k}^{N} a^{(k_1 k_2)}\rho^{k_1-1} z^{k_2}) + \cdots$$

$$z^{\cdot} = b^{(1,0)}\rho + b^{(2,0)}\rho^2 + b^{(1,1)}\rho z + b^{(0,2)}z^2 + \sum_{k_1+k_2=3}^{N} b^{(k_1 k_2)}\rho^{k_1} z^{k_2} + \cdots \tag{7.5}$$

If $b^{(1.0)} \neq 0$ in (7.5) then the problem of the stability of system (7.1) reduces to the problem of two roots equal to zero with one group of solutions.

In the case $b^{(1.0)} = 0$ the system (7.5) has two zero roots with two groups of solutions.

In both cases, it is indispensable to consider the variable ρ as being positive.

The investigation of the system (7.5) presents some complications only in the case in which all the coefficients $b^{(0.k)} (k = 1, \ldots, N)$ become equal to zero for any arbitrary large number N. In that case, the terms of order higher than the Nth on the right-hand side of the second equation of the system (7.5) do not have to become equal to zero for $\rho = 0$.

8. Let us consider this case. Going back to the system (6.1), we transform it by using $x = \xi + u(z, t), \quad y = \eta + v(z, t) \quad x_s = y_s + u_s(z, t) \qquad (s = 1, \ldots, n)$ which yields

$$\xi^{\cdot} = -\lambda\eta + \Xi(\xi, \eta, z, u, v, u_k, y_k, t), \qquad \eta^{\cdot} = \lambda\xi + H(\xi, \eta, z, u, v, u_k, y_k, t)$$

$$z^{\cdot} = Z^*(\xi, \eta, z, u, v, u_k, y_k, t), \qquad y_s^{\cdot} = \sum_{k=1}^{n} p_{sk} y_k + Y_s(\xi, \eta, z, u, v, u_k, y_k, t) \qquad (8.1)$$

$$(s, k = 1, \ldots, n)$$

The terms of order higher than the first, and not depending on $\xi, \eta, y_1, \ldots, y_n$ on the right-hand sides of that system appears in the form

$$\Xi(0, 0, z, u, v, u_k, 0, 0, \ldots, 0, t) = -\frac{\partial u}{\partial z} Z(u, v, z, u_k, t) - \lambda v + X(u, v, z, u_k, t) - \frac{\partial u}{\partial t}$$

$$H(0, 0, z, u, v, u_k, 0, \ldots, 0, t) = -\frac{\partial v}{\partial z} Z(u, v, z, u_k, t) + \lambda u + Y(u, v, z, u_k, t) - \frac{\partial v}{\partial t}$$

$$Z^*(0, 0, z, u, v, u_k, 0, \ldots, 0, t) = Z(u, v, z, u_k, t) \qquad (8.2)$$

$$Y_s(0, 0, z, u, v, u_k, 0, \ldots, 0, t) = -\frac{\partial u_s}{\partial z} Z(u, v, z, u_k, t) + \sum_{k=1}^{n} p_{sk} u_k + Y_s(u, v, z, u_k, t) - \frac{\partial u_s}{\partial t}$$

The coefficients $b^{(0.k)} (k = 2, \ldots, \infty)$ become equal to zero only when

$$Z(u, v, z, u_1, \ldots, u_n; t) = Z^*(0, 0, z, u, v, u_k, 0, \ldots, 0, t) \equiv 0.$$

Let us determine the values of the functions u, v, u_s from (8.2), under the condition

$$Z(u, v, z, u_k, t) = \Xi(0, 0, z, u, v, u_k, 0, \ldots, 0, t) = H(0, 0, z, u, v, u_k, 0, \ldots, 0, t) = \\ = Y_s(0, 0, z, u, v, u_k, 0, \ldots, 0, t) \equiv 0$$

then the right-hand sides of the system (8.1) become identically equal to zero if we set

$$\xi = \eta = y_1 = \ldots = y_n = 0$$

If the system (8.1) is transformed into the form (7.1), the forms $Z^{(l)}$ become zero when $r = 0$ for any l, no matter how large l is chosen. The second order system (7.4) corresponding to the system (8.1) is such that the straight line $\rho = 0$ is a singular line for the forms $R^{(l)}$ and $Z^{(l)}$ of any arbitrary high order. The investigation of these systems is complicated only when the forms $R^{(l)}$ and $Z^{(l)}$ determine stable motions for any arbitrary large l. It is obvious that the stability can be only of a nonasymptotic nature.

If,however, it is found that the unperturbed motion is unstable on the account of the forms of the ℓth order ($\ell \leq N$), then the integrals of the system (8.1) are also unstable.

Let us point out that if λ is rational, then the stability problem, represented by the system of equations (6.1) according to the method exposed in Section 4, is reduceable to the analysis of three equations having three roots and three groups of solutions.

If, however, the right-hand sides of the system (6.1) do not depend on time, then the system (6.1) takes the form (7.1) both for irrational and rational λ, the terms of the order higher than the Nth on the right-hand side of that system do not depend on t. In that case the solution (6.6) is periodic with a period $2\pi/\lambda$.

Note 8.1. We take advantage of this opportunity to point out that the method by which the investigation of the stability of a system of the $(n+2)$nd order is reduced to that of a second order system having the same critical variables, was first used in 1935 in the solution of Liapunov's problem (for two zero roots with one group of solutions) in [8]. The possibility of such a reduction was proved in that paper.

In 1936, in [9], this reduction method was applied to the solution of the stability problem in the case of two zero roots with two groups of solutions. In that paper it is proved that the stability and instability of the complete $(n+2)$nd order system, follows from the investigation of the shortened second order system.

In the 1939 paper [3], the reduction of systems of the $(m+2q+p)$th order is considered for steady state and periodic motions in nonessentially singular cases. The general statement concerning their reduction, and their investigation by means of systems of the $(m+2q)$th order is proved. In that work, the system of equations takes a form such that the search for Liapinov or Chetaev functions for the complete system reduces to the search for such functions for the shortened system. Since Liapunov's and Chetaev's theorems are reversible, the formulated statement is equivalent to the following. If the shortened system is asymptotically stable or unstabble, which follows from the consideration of the N first forms of the shortened system, independently from the forms of higher order, then, the complete system is correspondingly asymptotically stable or unstable.

The "reduction principle" refers here to the transformation of the given system into a form for which the functions of Liapunov ot Chetaev are constructed on the basis of the N first forms of the shortened system, and Liapunov and Chetaev functions for the complete system have the form

$$V = V_1(y_1, \ldots, y_{n_1}) + V_2(z_1, \ldots, z_p)$$

where V_1 is the Liapunov or Chetaev function corresponding to the shortened system, and V_2 is a quadratic form determined from Equation

$$\sum_{j=1}^{p} \frac{\partial V_2}{\partial z_j}(p_{j1}z_1 + \cdots + p_{jp}z_p) = \pm \sum_{j=1}^{p} z_j^2$$

Let us note that, when investigating critical cases, Liapunov always sought functions V, corresponding to the complete system. As a consequence of that, the ensemple of the terms containing $z_1, \ldots, z_p$ linearly was transformed into zero by an appropriate choice of the functions V and not by a transformation of the equations. In the simplest case of one zero root and a pair of pure imaginary roots, the reduction principle does not have a tangible superiority over the method used by Liapunov. However, in the case of two zero roots with one group of solutions, the complications were such that

Liapunov was forced to ignore the search for V functions and seek a solution to the problem in the form of series [10].

In [11] Malkin made an attempt to generalize the reduction principle to systems (1.1) for which the $p_{ji}(t)$ and $q_{sk}(t)$ are any arbitrary and bounded functions of t for $t \geq 0$. This generalization is based on a theorem published in [12]. In the proof of that theorem, Malkin has made an important error. It is asserted that with respect to the series $z_j = z_j(x_1, \ldots, x_n; t)$, satisfying Equations

$$\frac{\partial z_j}{\partial t} + \sum_{s=1}^{n} (p_{s1}x_1 + \cdots + p_{sn}x_n + X_s)\frac{\partial z_j}{\partial x_s} = Z_j(t; x_1, \ldots, x_n; z_1, \ldots, z_p)$$

the series $v_j = v_j(x_1, \ldots, x_n)$, determined by the system

$$\sum_{s=1}^{n} (\alpha_{s1}x_1 + \cdots + \alpha_{sn}x_n + Y_s)\frac{\partial v_j}{\partial x_s} = V_j(x_1, \ldots, x_n; v_1, \ldots, v_p) \qquad (j = 1, \ldots, p)$$

are intensifying if Y_s and V_j are obtained from X_s and Z_j by replacing the coefficients of their expansions by the largest moduli, and if the coefficients $\alpha_{s\sigma}$ for $\sigma < s$, represent the superior bounds of the moduli $p_{s\sigma}$. All the $\alpha_{ss} = \alpha$. This assertion is wrong. It is easy to convince oneself that the series $v = c_2x^2 + c_3x^3 + \ldots$, satisfying Equation

$$\frac{\partial v}{\partial x}(x + x^2) = x^2 + xv$$

is not stronger for the series $z = a_2x^2 + a_3x^3 + \ldots$, determined from Equation

$$\frac{\partial z}{\partial x}(-x - x^2) = x^2 + xz$$

since $a_2 = -\frac{1}{2}$; $c_2 = \frac{1}{2}$, but $c_3 = -\frac{1}{6}$ and $a_3 = \frac{1}{2}$. Thus the theorem on which is based the proof of the first basic theorem of the stability in critical cases is not proved. Consequently, the first basic theorem is not proved either.

In the proof of the deduction principle, Malkin used, without justification, the transformation given in [3]. The convergence of the series $z_j = z_j(x_1, \ldots, x_n; t)$ is not proved by Malkin, whereas the series $z_j = z_j(x_1, \ldots, x_n)$, appearing in [3] are absolutely convergent. Realizing the logical insufficiency of the discussions in the proof of this theorem, Malkin makes another attempt to prove the given theorem in [6], by using transformations different from those used by him in [11]. The proof, given in [6] contains an important error pointed out by Erugin [13]. The transformation

$$x_s = r^N \xi_s, \quad r = \sqrt{y_1^2 + \cdots + y_{n_1}^2}$$

can be used under the condition that the new variables ξ_s vary in the interval $\pm\infty$. Malkin considers $|\xi_s|$ and $|x_s|$ sufficiently small.

If the reduction principle is understood as it is formulated in [3], then the results concerning the systems (1.1) with constant and periodic coefficients, can be easily generalized to systems of the same form having coefficients continuous and bounded in t.

Let us assume that in the system (1.1) the coefficients $p_{ji} = 0$, $q_{sk} = 0$ for $i > j$ and $k > s$. This assumption does not decrease the generality of the problem [14]. Let the coefficients p_{jj} and q_{ss} satisfy the condition

$$\left|\left[\exp\int_0^t \left(p_{jj} - \sum_{s=1}^{n_1} k_s q_{ss}\right)dt\right]\int_0^t \exp\left[-\int_0^t \left(p_{jj} - \sum_{s=1}^{n_1} k_s q_{ss}\right)dt\right]dt\right| < M \qquad (A)$$

$$(k_1 + \cdots + k_{n_1} \leqslant N)$$

Then by means of the transformation

$$z_j = \zeta_j + \sum_{k=1}^{N} u_j^{(k)}(y_1, \ldots, y_{n_1}; t)$$

where $u_j^{(k)}(y_1, \cdots, y_{n_1}; t)$ are forms of the kth order in $y_1, \cdots, y_{n_1}$, and by a corresponding choice of those forms, the system (1.1) can be transformed in a form in which the functions playing the role of $Z_j^{(k)}(y_1, \cdots, y_{n_1}; t)$, become identically equal to zero for all $k \leq N$, when N is arbitrarily large. The coefficients of the forms $u_j^{(k)}$ under the conditions (A), are bounded and continuous. Assuming that the system (1.1) satisfies that condition, we write

$$y_s = \eta_s + \sum_{K \geqslant 1}^{N} u_s^{(k_1 \ldots k_{n_1})}(z_1, \ldots, z_p; t)\, y_1^{k_1} \ldots y_{n_1}^{k_{n_1}} \qquad (K = k_1 + \cdots + k_{n_1})$$

where the $u_s^{(k_1 \ldots k_{n_1})}$ are linear forms of $z_1, \ldots, z_p$.

The linear forms $u_s^{(k_1 \ldots k_{n_1})}(z_1, \ldots, z_p; t)$ can be determined such that the functions, playing the role of functions $P_s^{(k_1 \ldots k_{n_1})}$ for all $k_1 \ldots k_{n_1}$, satisfying the condition $k_1 + \ldots + k_{n_1} \leqslant N$. vanish in the transformed system. Similarly, the coefficients of the forms $u_s^{(k_1 \ldots k_{n_1})}$ are bounded and continuous.

If the system of equations (1.1) satisfying the condition (A), has coefficients p_{ji} such that the system of equations

$$z_1 = p_{11} z_1,\ z_2^{\cdot} = p_{21} z_1 + p_{22} z_2,\ \ldots,\ z_p^{\cdot} = p_{p1} z_1 + \cdots + p_{pp} z_p$$

has a Liapunov function V_2 of a quadratic form and satisfying the asymptotic stability theorem, then the Liapunov or Chetaev function for the complete system can be determined in the form

$$V = V_1(y_1, \cdots, y_{n_1}; t) + V_2(z_1, \cdots, z_p; t)$$

where V_1 is the Liapunov or Chetaev function for the shortened system.

It is necessary to note that the determination of the stability and instability by means of the N first forms of the right-hand sides of the "shortened" system, independently from the forms of higher order, as it was assumed by Malkin in [6], is more general than the method proposed in [3] which derives from the criteria of stability and instability as determined by functions of Liapunov and Chetaev. The determination of Malkin considers, in particular, the case of the nonasymptotic stability. However, Malkin does not clarify whether systems of equations having such property of motions for $g_{sk} = 0$ can exist or not. It can be asserted that for steady state and periodic motions, the systems of equations corresponding to the perturbed motion do not have such a property at least for $n_1 \leq 2$.

The example of Persidskii, presented in [6], refers to equations whose right-hand sides have linear terms.

Note 8.2. As was mentioned before, one can find in [3] the transformation of the system of equations (2.21) into equations with constant coefficents up to forms of any arbitrary high order, when there are m roots equal to zero, with m groups of solutions, and when all the λ_s are irrational and $\Sigma m_s \lambda_s \neq 0$. In [6], Malkin touches also this problem, considering λ_s as being irrational. In [6] the reduction is done without decreasing the order of the system.

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